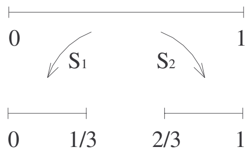
Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

ATTRACTOR CONSTRUCTION USING CONTRACTIONS

The Cantor Middle Thirds Set D can be constructed as the unique compact invariant set of two contractions.



$D_o = I = [\,0, 1\,]$

$D_1 = S(D_o)$

$S_1, S_2 : I \circlearrowleft$

$D_2 = S^2(D_o)$

$S_1(x) = \frac{x}{3}$, $S_2(x) = \frac{x}{3} + \frac{2}{3}$, $x \in I$.

Define $K(I) := \{K \subset I: K \neq \emptyset, K \text{ closed}\}$ and $S: K(I) \circlearrowleft$ by $S(K) = S_1(K) \cup S_2(K) \in K(I)$.

Then $D_1 = S(D_o)$, $D_2 = S(D_1) = S^2(D_o)$, and $D_n = S^n(D_o)$ $\forall n \in \aleph_o$. Thus, $D = \bigcap_0^\infty S^n(D_0)$.

D_n is composed of 2^n disjoint closed subintervals of I each of length 3^{-n}. Thus, D has zero Lebesque measure λ, because $\lambda(D) \leq (2/3)^n$ for all n.

Furthermore, D is **invariant** under S, i.e. S(D) = D, implying that D is a fixed point under S. This follows from the fact that

$$D = \{\ x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} = .\,a_1a_2a_3\ldots, \quad a_n \in \{0,2\}\ \}.$$

Proof:
It will be shown later that D has Hausdorff dimension ln2/ln3, which is about 0.6309.

Example: The Cantor Middle Thirds Set D

For $D := \bigcap_0^\infty D_n$, $S(D) = D$.

Proof: "$\subset$" $\quad S(D) \subset \bigcap_0^\infty S(D_n) = \bigcap_0^\infty D_{n+1} = D$, since $D_1 \subset D_o$.

"$\supset$" For $x \in D \cap [\,0, 1/3\,]$, $x = \sum_{n=2}^{\infty} \frac{a_n}{3^n} = .0a_2a_3\ldots$, with $a_n \in \{0,2\}$. Then, $x = S_1(y)$ for $y = .a_2a_3\ldots = \sum_1^\infty \frac{a_{n+1}}{3^n}$, i.e. $S_1(D) = D \cap [\,0, 1/3]$. Analogously, $S_2(D) = D \cap [\frac{2}{3}, 1] \Rightarrow S(D) = D$.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Remark

If K(I) is a complete metric space and if S is a contraction, S has exactly one fixed point D, the Cantor Middle Thirds Set, by the Contraction Mapping Principle and $D = \lim S^k(B)$, $B \in K(I)$. Thus, D is completely characterized as the unique compact invariant set of the contractions S_1 and S_2.

Definition

Let (X, d) be a metric space. A self map

$$S: X \circlearrowleft$$

is a *contraction* on X if there is a constant $0<c<1$:

$$d(S(x), S(y)) \leq c \cdot d(x, y) \qquad \forall x, y \in X;$$

c is called the *contractivity. If*

$$d(S(x), S(y)) = c \cdot d(x, y) \qquad \forall x, y \in X,$$

S is called a *similarity.*

Remarks

(1) A contraction is Lipschitz continuous, and therefore uniformly continuous.

(2) A similarity is injective, and if $K \subset X$ is compact, S induces a map

$$K \rightarrow S(K)$$

which is a homeomorphism. Thus, S transforms K into a geometrically similar copy of itself.

(3) If $S(x) = x$ and $S(y) = y$ for a contraction, then $x = y$, i.e. there is at most one fixed point for a contraction.

Examples

(1) The linear transformation

$$S: \mathfrak{R}^2 \circlearrowleft \qquad S(x,y) := (cx, cy) \quad \text{with } 0<c<1$$

is a similarity for the metric induced by the maximum norm

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

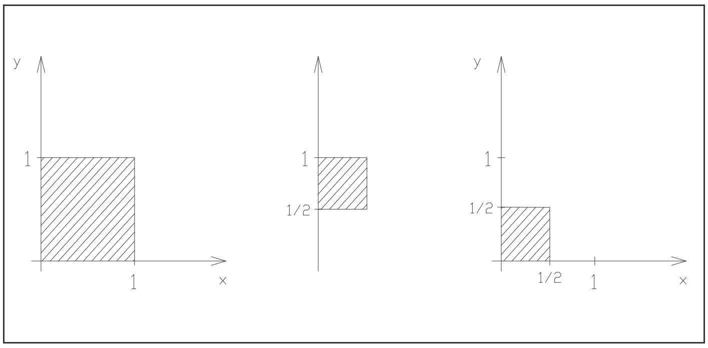
$|(x, y)|_m \quad := \quad \max(|x|, |y|).$

Proof : $S(x_1, y_1) - S(x_2, y_2) = (c \cdot x_1, c \cdot y_1) - (c \cdot x_2, c \cdot y_2)$

$= c \cdot (x_1 - x_2, y_1 - y_2)$ and

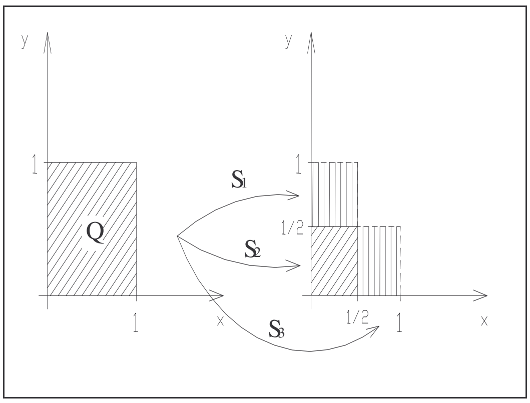
$|c \cdot (x_1 - x_2, y_1 - y_2)|_m = c \cdot |(x_1 - x_2, y_1 - y_2)|_m.$

(2) Let $S_1 := S_2 + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ with S_2 as in (1) for $c = ½$.



Q S1(Q) S2(Q)

(3) Let $S_3 := S_2 + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ with S_2 as in (1) for $c = ½$.



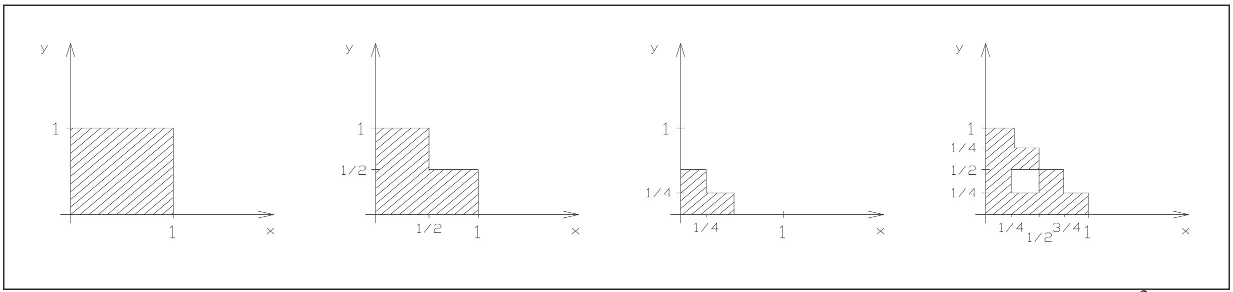
(4) Let Q be the closed unit square and

$K(Q) := \{K \subset Q: K \neq \emptyset, K \text{ closed}\}.$

Define a map $S := K(Q) \circlearrowleft$ by $S(K) := S_1(K) \cup S_2(K) \cup S_3(K)$ with S_1, S_2, S_3 as above

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

in (2) and (3).



Q S(Q) = Q_1 $S_2(Q_1)$ $S(Q_1)$ = S^2 (Q) = Q_2

Let Q_1 := S(Q). Then $Q_1 \subset Q$ and $Q_2 := S(Q_1) = S^2(Q) \subset Q_1$.

Note that the closed unit interval on the x- and the y -axis remain in $S^n(Q)$ for all n.

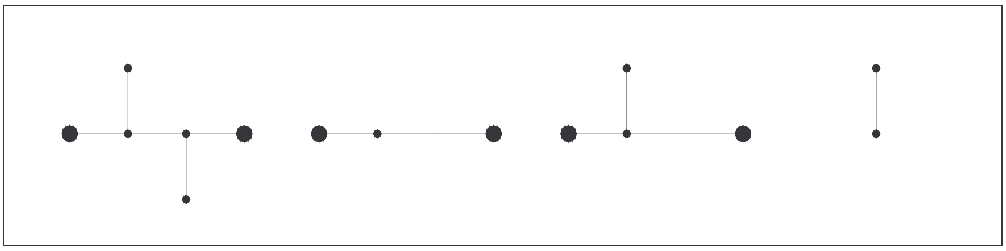
The nonempty compact set $\bigcap_0^\infty S^n(Q)$ is called the **Sierpinski Right Triangle**.

Remark: A decreasing sequence $K_n \supset K_{n+1}$ of nonempty compact sets in a topological space has a nonempty compact intersection K .

Proof: $K \subset K_1$ and K is closed, thus compact. Because the finite intersection of the K_n is nonempty, so is K by the **finite intersection property of compact sets**.

Example

In the following 4 sketches, each sketch is composed of line segments. Each segment defines a similarity by mapping the 2 big dots onto the end points of the segment.



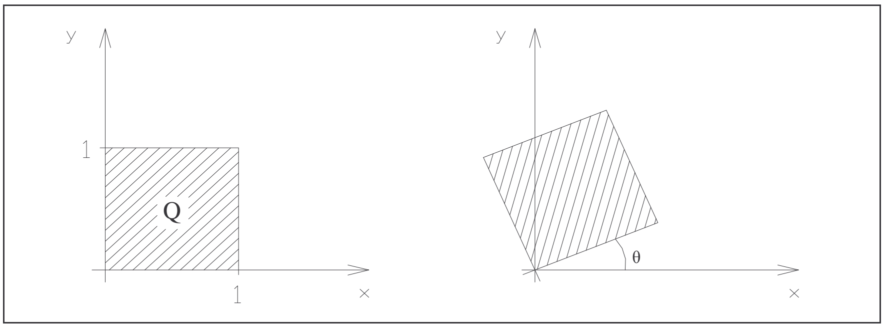
For example, the second figure represents two similarities, a reduction by 1/3 in the x direction, i.e.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

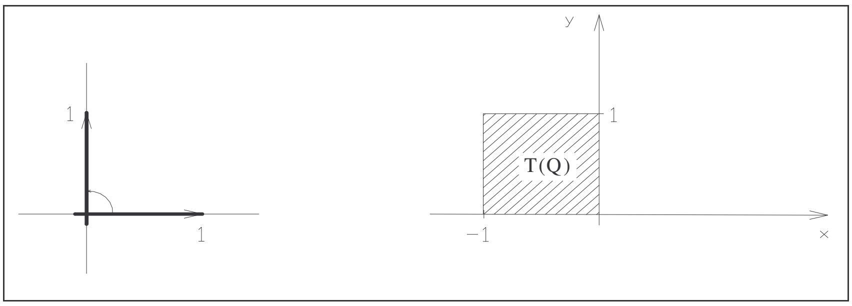
$S_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/3 \\ y \end{pmatrix}$, and the composition of a reduction by 2/3 in the x direction followed by a translation by (1/3, 0). The third figure represents an additional similarity being included, namely a reduction by 1/3 in the x direction then translating by (1/3,0) and rotating by $\pi/2$ counterclockwise if (1/3,0) is taken as the origin.

Note that a **rotation by θ counterclockwise** is the linear transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \qquad A := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$



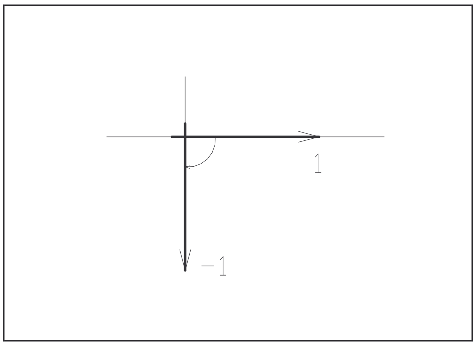
For $\theta = \frac{\pi}{2}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$.



A rotation by $\frac{\pi}{2}$ counterclockwise is an isometry, i.e. $|T(x,y)|_m = |(x,y)|_m$.

Note that a rotation by $-\frac{\pi}{2}$ clockwise is given by $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de



Definition

Let X be a metric space. A finite set of contractions $S_j: X \circlearrowleft$, $1 \le j \le n$, is called *an iterated function scheme*, abbreviated IFS (which is a notation coined by Barnsley and Demo in 1985 [B/D]). Let

$$\mathcal{K}(X) := \{K \subset X : K \neq \emptyset, K \text{ compact}\},$$

and define

$$S := \mathcal{K}(X) \circlearrowleft \quad \text{by} \quad S(K) := \bigcup_{j=1}^{n} S_j(K).$$

Then $K \in \mathcal{K}(X)$ is *invariant* if $S(K) = K$.

The fundamental theorems concerning finitely many contractions were proved by J. Hutchinson in 1981 in the paper “Fractals and Similarities”, Indiana Univ. J.Math. **30**, 713-747. In this paper Hutchinson developed the mathematical background for Mandelbrot’s visionary book “The Fractal Geometry of Nature” , Freeman, 1977.

Theorem 1 (Hutchinson, 1981 (3.1))

Let X be a complete metric space. There is a unique nonempty compact invariant set $A \subset X$ associated to finitely many contractions $S_j: X \circlearrowleft$, $1 \le j \le n$; A is called the *attractor* for S.

For any $B \in \mathcal{K}(X)$

$$A = \lim_{k \to \infty} S^k(B)$$

with S^k the k-fold composition of S with itself. The convergence is with respect to the Hausdorff

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

metric d_H on K(X) which will be defined shortly.

If $S(B) \subset B$, then

$$A = \bigcap_{1}^{\infty} S^k(B) .$$

Remarks

1) In the first part of Theorem 1 the set B can even be a singleton, i.e. a point in X.

2) The proof is based on the fact that there is a natural metric making K(X) a complete metric space and S a contraction; the Contraction Mapping Principle then implies the first part of 2.1. This metric will be introduced now.

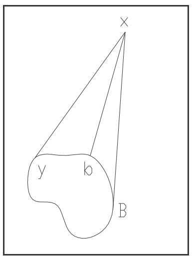
Hausdorff metric on K(X), X a metric space

Let (X, d) be a metric space. Then d can be used to define the distance between subsets $A, B \subset X$:

Definition

If $B \subset X$ is nonempty and $x \in X$, define

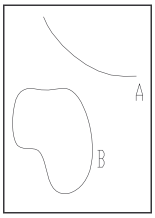
$$d(x, B) := \inf_{y \in B} d(x, y) \geq 0.$$



If A is also nonempty, define

$$d(A, B) := \inf d(x, B) \geq 0, \text{ where the infimum is taken over all } x \text{ in } A.$$

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de



Remarks

(1) $d(x, B) \in [0,\infty)$, since $\{d(x, y): y\in B\} \neq \emptyset$.

(2) If B is compact, there is a closest point in B to any point $x\in X$:

$$d(x, B) = d(x, b) \text{ for at least one point } b\in B.$$

<u>Proof:</u>

$d: X \times X \rightarrow \Re$ is continuous $\Rightarrow$ $d(x, \cdot) : X \rightarrow \Re$, $y \rightarrow d(x, y)$, is continuous.

Since $B\subset X$ is compact, $d(x, \cdot)$ has a minimum in B,

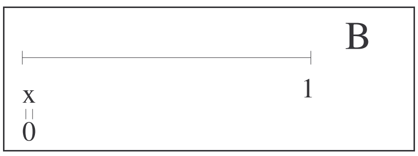
i.e. $\exists\, b \in B$: $d(x, b) = \min_{y\in B} d(x, y) = d(x, B)$.

(3) $d(x, B) = 0 \Leftrightarrow x \in \overline{B}$. This implies that $d: X\times X \rightarrow \Re$ does **not** induce a metric on the set P(X) of all subsets of X.

Proof: $x\in \overline{B} \Leftrightarrow x = \lim b_n$

Example

$X = \Re$, $x = 0$, $B = (0, 1)$. Then
$d(0, (0,1)) = 0$ but $\{0\}\cap (0, 1) = \emptyset$.

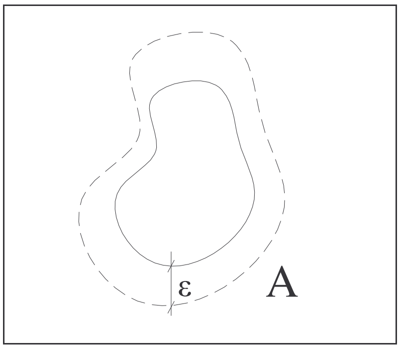


Definition

Let (X, d) be a metric space and $\varepsilon>0$. The (open) ε – *neighborhood* of $A\subset X$ is

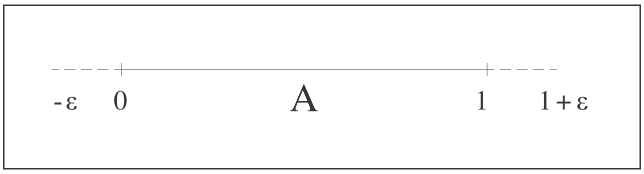
Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

$A_\varepsilon := \{x \in X : \exists\, a \in A \text{ with } d(x, a) < \varepsilon\}$, i.e. A is thickened by ε.



Examples

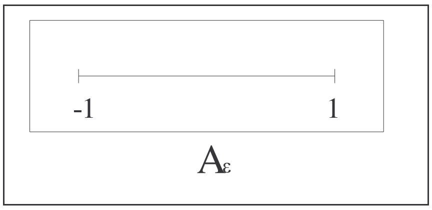
(1) $X = \Re$, $A = (0, 1)$, $\varepsilon > 0 \Rightarrow A_\varepsilon = (-\epsilon, 1 + \varepsilon)$.



(2) $X = \Re^2$, $A = B_1(0)$ the open disc with center 0 and radius 1, $\varepsilon > 0 \Rightarrow A_\varepsilon$ is the open disc with center 0 and radius $1+\varepsilon$

(3) $X = \Re^2$, $A = (-1, 1)$, $\varepsilon > 0 \Rightarrow A_\varepsilon$ is the open rectangle of length $2 + 2\varepsilon$ and height 2ε with

$d = |\ |_m$.



Remarks

(1) $A_\varepsilon = \bigcup_{a \in A} B_\varepsilon(a) \supset A$, where $B_\varepsilon(a)$ is the ball around a point a in A with radius ε.

(2) $A \subset B \Rightarrow A_\varepsilon \subset B_\varepsilon$.

(3) $r < s \Rightarrow A_r \subset A_s$.

(4) $B = \bigcap_{\varepsilon > 0} B_\varepsilon$ if B is closed

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Proof:

"$\subset$" is trivial, and if $x \in B_{1/n}$ $\forall n \in \aleph$, then for every n there is a $b_n \in B$ with $d(b_n, x) < \frac{1}{n}$ $\Rightarrow$ $b_n \to x$ $\Rightarrow x \in B$, because B is closed.

(5) $\alpha > 0$, $\beta > 0$ $\Rightarrow$ $(A_\beta)_\alpha \subset A_{\alpha+\beta}$

Proof:

$x \in (A_\beta)_\alpha$ $\Rightarrow$ $\exists\ y \in A_\beta$ with $d(x, y) < \alpha$ $\Rightarrow$ $\exists\ a \in A$ with $d(y, a) < \beta$

The Triangle Inequality implies $d(x, a) < \alpha + \beta$.

(6) For a contraction S: X⮌ with contractivity c and $B \subset X$,

$$S(B_\varepsilon) \subset (S(B))_{c\varepsilon} \text{ for every } \varepsilon > 0.$$

Proof: $\forall\ x \in B_\varepsilon\ \exists\ b \in B$: $d(x, b) < \varepsilon$

$\Rightarrow d(S(x), S(b)) \le c \cdot d(x, b) < c\varepsilon$

Definition

If A, B $\subset$ X, define

$$D(A, B) := \inf\{\varepsilon > 0 : A \subset B_\varepsilon, B \subset A_\varepsilon\} \in [0, \infty]$$

whereby $\inf \emptyset := \infty$.

Examples

(1) $X = \Re$, $A = \emptyset$, $B = \{0\}$ $\Rightarrow$ $D(\emptyset, \{0\}) = \infty$.

Proof:

$A = \emptyset \subset B_\varepsilon$ $\forall\ \varepsilon$ and $A_\varepsilon = \emptyset\ \forall\ \varepsilon$. However, for every ε, $B \not\subset A_\varepsilon = \emptyset$, since $B \neq \emptyset$.

In general, if B is a subset of X and if $B \neq \emptyset$, then $D(\emptyset, B) = \infty$, i.e. D can be used to define a metric **at most** on the set $\{A \subset X : A \neq \emptyset\} \subset P(X)$.

(2) $X = \Re$, $A = \{0\}$, $B = [0, \infty)$ $\Rightarrow$ $D(A, B) = \infty$.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Proof:

$A \subset B \;\Rightarrow\; A \subset B_\varepsilon\ \forall\ \varepsilon$ but there is no $\varepsilon>0$ with $B \subset A_\varepsilon$. Therefore,

D can be used to define a metric **at most** on the set $\{A \subset X: A \neq \emptyset$, A *bounded*$\} \subset P(X)$.

(3) $X = \Re$, $A = (0, 1)$, $B = [0, 1] \;\Rightarrow\; D(A, B) = 0$ (but $A \neq B$).

Proof:

$A \subset B \Rightarrow A \subset B_\varepsilon\ \forall\ \varepsilon$.

$B \subset A_\varepsilon\ \forall \varepsilon$ and $\inf\{\varepsilon>0\} = 0$.

Remark

D defines a metric **at most** on the set

$$B(X) := \{A \subset X: A \neq \emptyset, \text{A closed and bounded}\}$$

which contains K(X).

Theorem 2

If X is a metric space, then

$$d_H: K(X) \times K(X) \to \Re, \qquad (A, B) \to d_H(A, B) := \inf\{\varepsilon>0: A \subset B_\varepsilon, B \subset A_\varepsilon\}$$

defines a metric on $K(X) := \{A \subset X: A \neq \emptyset$, A compact$\}$ called the *Hausdorff metric* .

Proof:

(1) d_H is well-defined:

$d_H(A, B) \in [0, \infty)$, since A and B are compact .
$d_H(A, B) = \max\{d(x, B) : x \in A\}$.

(2) Symmetry:

$d_H(A, B) = d_H(B, A)$ is obvious.

(3) $d_H(A, B) = 0 \Leftrightarrow A = B$:

Proof:
"$\Leftarrow$" Follows from $\inf\{\varepsilon>0\} = 0$.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

"⇒" If $\varepsilon > 0$, then $A \subset B_\varepsilon$ and $B \subset A_\varepsilon$.

$$\Rightarrow A \subset \bigcap_{\varepsilon>0} B_\varepsilon = B \text{ and } B \subset \bigcap_{\varepsilon>0} A_\varepsilon = A .$$

(4) Triangle Inequality: $d_H(A, B) \le d_H(A, C) + d_H(C, B)$:

Proof:

Let $\alpha > d_H(A, C)$, $\beta > d_H(C, B)$.

It will be shown that $\alpha + \beta \ge d_H(A, B)$:

$A \subset C_\alpha$, $C \subset B_\beta \Rightarrow A \subset (B_\beta)_\alpha \subset B_{\alpha+\beta}$.

Similarly, $B \subset A_{\alpha+\beta} \Rightarrow \alpha + \beta \ge d_H(A, B)$.

With $\alpha = d_H(A,C) + 1/n$, $\beta = d_H(C, B) + 1/n$, the Triangle Inequality (4) follows.

Remarks

(1) $K_n \to K$ in $(K(X), d_H)$ iff $\forall \varepsilon > 0 \; \exists N \in \aleph : \; d_H(K_n, K) < \varepsilon \; \forall n \ge N$.

(2) $d_H(A, B) = \max\{\inf\{\varepsilon>0: B \subset A_\varepsilon\}, \inf\{\varepsilon>0: B \subset A_\varepsilon\}\}$.

Proof:
Let $m = \max\{\inf\{\varepsilon>0: B \subset A_\varepsilon\}, \inf\{\varepsilon>0: A \subset B_\varepsilon\}\}$. Because

$$\{\varepsilon>0: A \subset B_\varepsilon,\ B \subset A_\varepsilon\} \subset \{\varepsilon>0: A \subset B_\varepsilon\},$$

then $d_H(A, B) \ge m$. If $d_H(A, B) > m$, then $\exists\, \varepsilon > 0$ with $\varepsilon < d_H(A, B)$, $B \subset A_\varepsilon$

and $A \subset B_\varepsilon$, implying the contradiction $\varepsilon \ge d_H(A, B)$. Thus, $d_H(A, B) = m$.

(3) $d_H(A, B) = \max\{ d(x, B) : x \in A \} = \max\{ d(A, x) : x \in B \}$
Proof:

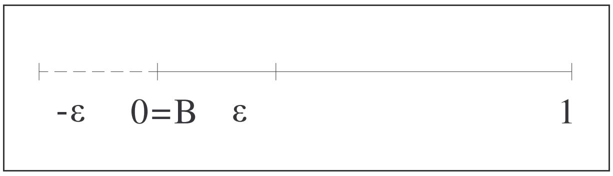
(4) $d(A, B) \le d_H(A, B)$, i.e. the Hausdorff distance is at least equal to the Euclidean distance.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Examples

(1) $X = \Re$, $A = [0, 1]$, $B = \{0\} \Rightarrow d_H(A, B) = 1$.

Proof:



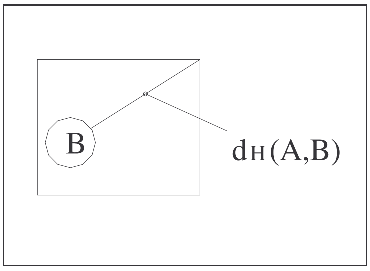
$B \subset A \subset A_\varepsilon \; \forall \varepsilon$

$A \subset B_\varepsilon = (-\varepsilon, \varepsilon) \Leftrightarrow \varepsilon > 1$.

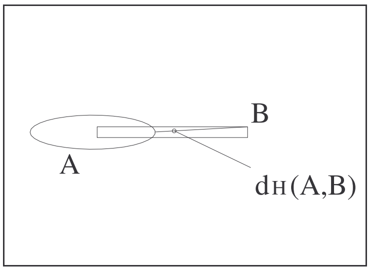
$d_H(A, B) = \inf\{\varepsilon > 1\} = 1$.

Note: $d(0, [0, 1]) = 0$, i.e. the Euclidean distance is 0, but the Hausdorff distance is 1.

(2) $X = \Re^2$



(3) $X = \Re^2$



Remark

If X is a metric space, then $K \subset X$ is compact if and only if every sequence in K contains a subsequence which converges in K (**Bolzano - Weierstraß** Property).

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Theorem 3 (Hutchinson)

If (X, d) is a complete metric space, then $(K(X), d_H)$ is complete.

Proof:

Let $(K_n)_{n\in\aleph}$ be a Cauchy sequence in K(X).

Define $K:=\{x\in X: \exists\, x_m \in K_m$, for every $m\in\aleph$ and $x_m \to x\}$.

(1) $\boxed{K_n \to K}$

Let $\varepsilon > 0$. Since $(K_n)_n$ is Cauchy, $\exists\, N\in\aleph$:

(*) $d_H(K_k, K_n) < \dfrac{\varepsilon}{2} \quad \forall\, k, n \geq N$.

It will be shown that $K\subset (K_n)_\varepsilon$ and $K_n \subset K_\varepsilon$ for $n \geq N$, implying $d_H(K_n, K) \leq \varepsilon$

$\forall\, n \geq N$. Let $n \geq N$ be fixed.

a) $\boxed{K \subset (K_n)_\varepsilon}$

For $x\in K\ \exists\, x_m\in K_m \forall\, m$ so that $x_m \to x \Rightarrow \exists\, M\in\aleph$:

$d(x_m, x) < \dfrac{\varepsilon}{2} \quad \forall m\geq M$.

If $k \geq N$, $\exists\, y\in K_n$ with $d(x_k, y) < \dfrac{\varepsilon}{2}$ by (*) .

$\Rightarrow d(x, y) \leq d(x, x_k) + d(x_k, y) < \varepsilon \quad \forall\, k \geq \max\{N, M\}$

$\Rightarrow K \subset (K_n)_\varepsilon$.

b) $\boxed{K_n \subset K_\varepsilon}$

Let $y\in K_n$. For $j \in \aleph$, set $\varepsilon_j := \dfrac{\varepsilon}{2^{j+2}}$.

Since $(K_n)_n$ is a Cauchy sequence, $\exists\, N_j \in \aleph$:

(**) $d(K_k, K_l) < \varepsilon_j \quad \forall\, l, k \geq N_j$

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

A sequence $n < N_1 < N_2 \dots$ and a sequence $x_{N_j} \in K_{N_j}$ satisfying the two inequalities

$$d(x_{N_j}, x_{N_{j+1}}) < \varepsilon_j \quad \text{and} \quad d(y, x_{N_j}) < (3/4)\varepsilon \quad \forall\, j \in \aleph$$

will be constructed. The first inequality insures that $(x_{N_j})_j$ is a Cauchy sequence in X , and since X is complete, it has a limit x in X. After the **Extension Lemma for Cauchy Sequences** (see below), there is a Cauchy sequence $(\tilde{x}_n)_{n \in \aleph}$ in X with $\tilde{x}_n \in K_n \ \forall\, n$ such that

$$\tilde{x}_{N_j} = x_{N_j} \qquad \forall j.$$

Then $x \in K$ follows. The second inequality above implies

$$d(y, x) = d(y, \lim_{j \to \infty} x_{N_j}) = \lim_{j \to \infty} d(y, x_{N_j}) < (3/4)\varepsilon < \varepsilon$$

$\Rightarrow$ $y \in K_\varepsilon$ and $K_n \subset K_\varepsilon$.

To begin the construction, let $N_1 > n$ satisfy (**) for j =1.

By (*), $K_n \subset (K_{N_1})_{\frac{\varepsilon}{2}} \Rightarrow \exists\, x_{N_1} \in K_{N_1}: \ d(y, x_{N_1}) < \frac{\varepsilon}{2} =: \varepsilon_0$.

Choose $N_2 > N_1$ satisfying (**) for j= 2. By (**) for j = 1 ($k = N_1, l = N_2$)

$$K_{N_1} \subset (K_{N_2})_{\varepsilon_1}$$

$\Rightarrow \exists\, x_{N_2} \in K_{N_2}$ with $d(x_{N_1}, x_{N_2}) < \varepsilon_1$

$$\Rightarrow d(y, x_{N_2}) \le d(y, x_{N_1}) + d(x_{N_1}, x_{N_2}) < \varepsilon_0 + \varepsilon_1 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2^3}$$

By induction, sequences $n < N_1 < N_2 < \dots$ and $x_{N_j} \in K_{N_j}$ are constructed so that $(x_{N_j})_j$ is a Cauchy sequence and

$$d(y, x_{N_j}) < \varepsilon_0 + \varepsilon_1 + \dots \varepsilon_{j-1} = \varepsilon\left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{j+1}}\right) < \varepsilon\left(\frac{1}{2} + \sum_{k=3}^{\infty} \frac{1}{2^k}\right) =$$

$$= \varepsilon\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{4}\,\varepsilon.$$

(2) $K \in K(X)$

$K \neq \emptyset$, since $\exists$ Cauchy sequence (x_n), $x_n \in K_n$, constructed as above.
K closed: Let $a_j \in K$, $j \in \aleph$: $a_j \to a \in X$. It will be shown that $a \in K$.

By definition of K, for every j there is a sequence $(x_{j,k})$ in K_k $\forall$ k with

(*) $$a_j = \lim_{k\to\infty} x_{j,k} .$$

Since $a_j \to a$,

$\exists N_1 < N_2 < \ldots$ such that $d(a_{N_j}, a) < \frac{1}{j}$.

Due to (*), there is a subsequence of integers $m_1 < m_2 < \ldots$ with

$$d(x_{N_j,m_j}, a_{N_j}) < \frac{1}{j} .$$

$$\Rightarrow d(x_{N_j,m_j}, a) \leq \frac{2}{j}.$$

Let $y_{m_j} := x_{N_j,m_j} \in K_{m_j} \Rightarrow y_{m_j} \to a$. By the **Extension Lemma for Cauchy Sequences**, a is in K.

K compact:

It will be shown that K is **totally bounded**, i.e. for every $\varepsilon > 0$
there are finitely many open balls with radius ε and centers in K covering K.

Assume the contrary. Then there is a positive ε such that no covering of K by finitely

many balls of radius ε with centers in K exists. Now a sequence (x_i) in K will be constructed with

(+) $$d(x_i, x_j) \geq \varepsilon \quad \text{for } i \neq j.$$

Choose any point x_1 in K. Since $B_\varepsilon(x_1)$ doesn't cover K, there is an x_2 in K with

$d(x_2, x_1) \geq \varepsilon$. If $x_1, \ldots, x_n$ are in K and satisfy (+), choose x_{n+1} in K with $d(x_{n+1}, x_j) \geq \varepsilon$

for $1 \leq j \leq n$. Using a) of part (1) of the proof, for every i there exists a point y_i in K_N with

$$d(x_i, y_i) < \varepsilon/3.$$

However, K_N is compact, and there must be a convergent subsequence (y_{i_j}) of (y_i) by the

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Bolzano-Weierstrass property, which is, of course, Cauchy. Thus, an M∈ ℵ exists with

$$d(y_{i_j}, y_{i_k}) < \varepsilon/3$$

for j, k ≥ M, implying that

$$d(x_{i_j}, x_{i_k}) < \varepsilon$$

for j ,k ≥ M, a contradiction to (+).

Note : Theorem 3 holds when K(X) is replaced by B(X).

Extension Lemma for Cauchy Sequences

If (X, d) is a metric space, let $(K_n)_{n\in ℵ}$ be a Cauchy sequence in $(K(X), d_H)$.Let

$$0 < n_1 < n_2 < \ldots$$

be a subsequence of integers. If $(x_{n_j})_{j\in ℵ}$ is a Cauchy sequence in X with $x_{n_j} \in K_{n_j}$ ∀ j, there is another Cauchy sequence $(\tilde{x}_n)_{n\in ℵ}$ in X with $\tilde{x}_n \in K_n$ ∀ n such that

$$\tilde{x}_{n_j} = x_{n_j} \qquad \forall j.$$

Proof:

Contractions on $(K(X), d_H)$ for a metric space (X, d)

Theorem 4

If (X, d) is a metric space, then every contraction S: X⮌ with contractivity c defines a contraction S: K(X)⮌ on $(K(X), d_H)$ with the same contractivity c by

$$S(A) := \{ S(x): x\in A\}$$

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

for $A \in K(X) := \{A \subset X: A \neq \emptyset, A \text{ compact}\}$.

Proof:

(1) $S: K(X) \circlearrowleft$ is well defined:

$S(A) \in K(X)$, because S is continuous and A is compact.

(2) $d_H(S(A), S(B)) \leq c\, d_H(A, B), \quad A, B \in K(X)$:

Let $\varepsilon > d_H(A, B)$. It will be shown that $d_H(S(A), S(B)) \leq c\varepsilon$.

Since $A \subset B_\varepsilon \Rightarrow S(A) \subset S(B_\varepsilon)$.

Now $S(B_\varepsilon) \subset (S(B))_{c\varepsilon} \Rightarrow S(A) \subset (S(B))_{c\varepsilon}$.

Similarly, $S(B) \subset (S(A))_{c\varepsilon}$. $\Rightarrow d_H(S(A), S(B)) \leq c\varepsilon \quad \forall \varepsilon > d_H(A, B)$.

If $\varepsilon_n := d_H(A, B) + \frac{1}{n}$, then $d_H(S(A), S(B)) \leq c \cdot d_H(A, B) + \frac{c}{n} \quad \forall n \in \aleph$.

Taking $\lim_{n\to\infty}$ gives (2).

Remark

$A_\varepsilon \cup B_\varepsilon \subset (A \cup B)_\varepsilon$ for $A, B \in K(X)$.

Proof:

$A, B \subset A \cup B \Rightarrow A_\varepsilon, B_\varepsilon \subset (A \cup B)_\varepsilon$.

Corollary 5 (Hutchinson, 1981 (3.2))

Finitely many contractions $S_j: X \circlearrowleft$ with contractivity c_j, $1 \leq j \leq n$, define a contraction $S: K(X) \circlearrowleft$ with contractivity $c := \max_{1 \leq j \leq n} c_j$ by

$$S(A) := \bigcup_{j=1}^{n} S_j(A)$$

for $A \in K(X)$.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Proof:

(1) S: K(X) ⮌ is well defined:

$S_j(A) \in K(X), \; 1 \leq j \leq n \;\Rightarrow\; S(A) \in K(X).$

Notice that the hypothesis "finitely" is used here.

(2) $d_H(S(A), S(B)) \leq \max_{1\leq j\leq n} d_H(S_j(A), S_j(B)) =: M$:

Note that the hypothesis "finitely" is also used here to insure the existence of such an $M \in \Re$.

Let $\varepsilon > M \;\Rightarrow\; S_j(A) \subset (S_j(B))_\varepsilon \;\forall j$. Therefore, $S(A) \subset \bigcup_{j=1}^{n} (S_j(B))_\varepsilon$

and it follows from the remark above that $S(A) \subset (S(B))_\varepsilon$.

Similarly, $S(B) \subset S(A)_\varepsilon$.

$\Rightarrow d_H(S(A), S(B)) \leq \varepsilon \quad \forall \varepsilon > M.$

For $\varepsilon_n := M + \frac{1}{n}$, $\; d_H(S(A), S(B)) \leq M + \frac{1}{n} \quad \forall n \in \aleph$.

Taking $\lim_{n\to\infty}$ gives (2).

(3) For $j \in \{1, \ldots, n\}$ with $M = d_H(S_j(A), S_j(B))$,

$d_H(S(A), S(B)) \leq M \leq c_j \cdot d_H(A, B) \leq c \cdot d_H(A, B).$

Remark

If $A, B, C \in K(X)$ with $A \subset B \subset C$, then $d_H(A, C) \geq d_H(A, B)$

Proof:

Let $\varepsilon > d_H(A, C) \;\Rightarrow\; C \subset A_\varepsilon \;\Rightarrow\; B \subset A_\varepsilon \;\Rightarrow\; \varepsilon \geq d_H(A, B).$

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Lemma 6

Let X be a metric space, and let $K_n \in K(X)$, $n \in \aleph$, be decreasing, i.e.

$$K_n \supset K_{n+1}, \quad n \in \aleph.$$

If $K \in K(X)$ with $K \subset K_n$, $n \in \aleph$, then

$$K = \bigcap_1^\infty K_n \quad \Leftrightarrow \quad \lim_{n\to\infty} K_n = K, \text{ i.e. } \quad d_H(K_n, K) \underset{n\to\infty}{\to} 0.$$

<u>Proof:</u> "$\Leftarrow$": $K \subset \bigcap_1^\infty K_n \subset K_n$. After the remark above, $0 \le d_H(K, \bigcap_1^\infty K_n) \le d_H(K, K_n)$ $\forall n \in \aleph \Rightarrow d_H(K, \bigcap_1^\infty K_n) = 0 \Rightarrow K = \bigcap_1^\infty K_n$.

"$\Rightarrow$" : $K := \bigcap_1^\infty K_n \subset K_{n+1} \subset K_n$.

Let $r_n := d_H(K_n, K) = \inf\{\varepsilon > 0 : K_n \subset K_\varepsilon\}$ for $n \in \aleph$. By the remark above,

$r_n \ge r_{n+1} \ge 0, \ n \in \aleph \Rightarrow \exists \ r \ge 0: \ r_n \to r.$

If $r > 0$, let $\varepsilon := r/2$. Because $r_n \ge r > \varepsilon$, it follows that $K_n \not\subset K_\varepsilon$,

and for every n there is an $x_n \in K_n$ such that $x_n \notin K_\varepsilon$.

Since $K_n \subset K_1$ for every n, the sequence (x_n) has a convergent subsequence $(x_{k_j})_j$ and thus a limit x in K_1 by the Bolzano-Weierstrass property. Then $\exists N \in \aleph$: $d(x_{k_j}, x) < \varepsilon \ \forall j \ge N$.

If $x \in K$, a contradiction to $x_{k_j} \notin K_\varepsilon$ follows, and r must vanish.

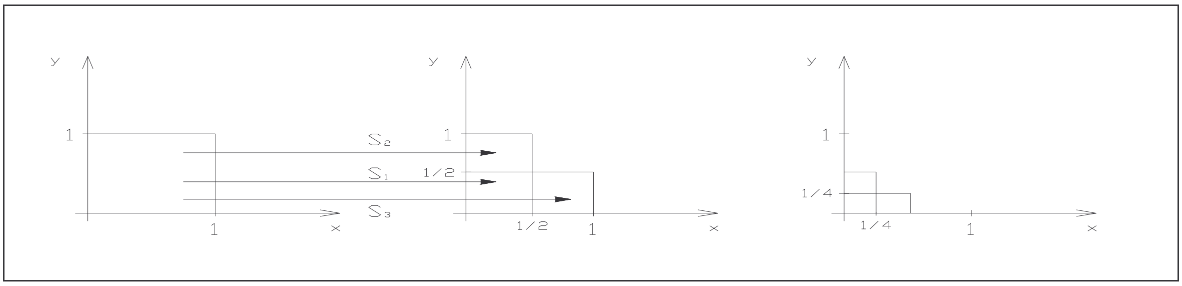
Fix $m \in \aleph$ and consider $j > m$. Then $k_j > k_m$ and $K_{k_j} \subset K_{k_m}$

$\Rightarrow \quad x_{k_j} \in K_{k_m} \ \forall j > m . \quad \Rightarrow \quad x \in K_{k_m}$ for all $m \in \aleph$ implying $x \in K$.

Proof of Theorem 2.1: After 2.3, 2.5 and 2.6, Theorem 2.1 follows using the Contraction Mapping Principle.

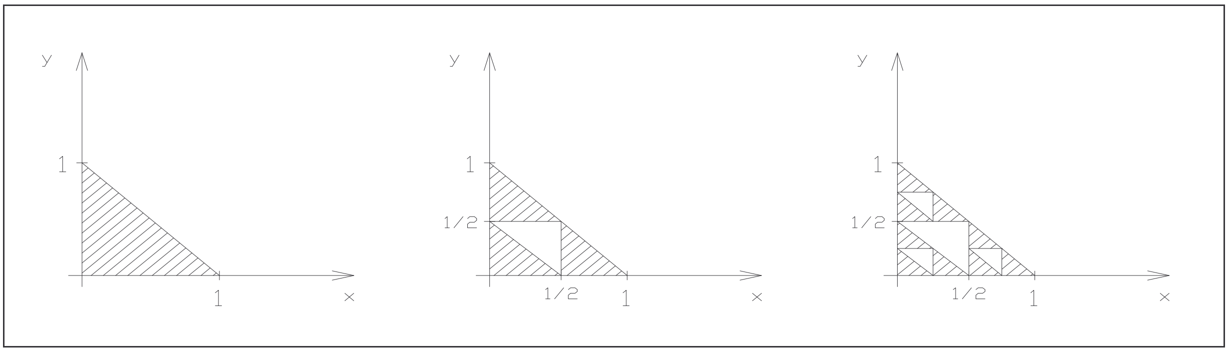
Example: **The Sierpinski Right Triangle**

The attractor for the 3 similarities in $\mathfrak{R}^2$ given by



is the Sierpinski Right Triangle (W. Sierpinski, 1915), and can also be defined recursively as follows using infinite removals.

Let D_o be a right triangle with side lengths 1 (hypotenuse $\sqrt{2}$). Divide D_o into 4 right triangles by joining the midpoints of each side. Each of the 4 smaller triangles has side length $\frac{1}{2}$ and one of these triangles is inverted.



D_0 **D_1** **D_2**

Define D_1 by removing the open inverted triangle, i.e. remove its interior but not its boundary. Proceed recursively: Define D_{n+1} by removing the open inverted triangle from each of the 3^n right triangles in D_n which have side lengths $\frac{1}{2^n}$.

Let D $:= \bigcap_0^\infty D_n$. Then D is nonempty and compact,

because $D_n \supset D_{n+1}$ for every n , and D_n is nonempty and compact.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Remark

The area of D is 0 , i.e. D has zero Lebesque measure λ(D) in $\Re^2$.

Step	*Area*	*Number of triangles*	*Side lengths*
D_o	$\frac{1}{2} = a$	1	1
D_1	$\frac{3}{4}a$	3	$\frac{1}{2}$
D_2	$\frac{3^2}{4^2}a$	3^2	$\frac{1}{2^2}$
…	…	…	…
D_n	$1/2 \left(\frac{3}{4}\right)^n a$	3^n	$\frac{1}{2^n}$

$$\lambda(D) \leq 1/2 \left(\frac{3}{4}\right)^n a \qquad \forall n \in \aleph \quad \Rightarrow \quad \lambda(D) = 0.$$

Remark

The boundaries of each triangle in D_n remain in D_{n+1} and therefore, D contains all of these line segments. At step n, 3^n new segments arise, each with a length of at least 2^{-n}. The length of all the segments in D_n is at least $3^n 2^{-n} \rightarrow \infty$, implying that D has infinite length, i.e. infinite Lebesque measure in $\Re$.

Later we will see that it has a dimension of about 1.58.

Remark

D is the attractor for the three contracting similarities

$$S_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix}, \qquad S_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2}+\frac{1}{2} \end{pmatrix}, \qquad S_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2}+\frac{1}{2} \\ \frac{y}{2} \end{pmatrix},$$

i.e. $D = \lim_{k\to\infty} S^k(B)$, $B \in K(\Re^2)$, and if $S(B) \subset B$, then

$$D = \bigcap_0^\infty S^k(B).$$

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Proof: The claim is that S(D) = D. It is immediate that $D_{n+1} = S(D_n)$ for every n, implying

$$S(D) = S\left(\bigcap_0^\infty D_n\right) \subset \bigcap_0^\infty S(D_n) = D.$$

To prove that $D \subset S(D)$, let p be in D. Then $p \in D_1 = S(D_o) = S_1(D_o) \cup S_2(D_o) \cup S_3(D_o)$.

It is no restriction to assume that p belongs to only **one** of the three subtriangles $S_j(D_0)$, since otherwise p is either (0,½), (½, ½), or (½,0), which is $S_1(q)$, $S_2(q)$ or $S_3(q)$ for q = (0,1), (1,0) or (0,0). Therefore, assume that p is in $S_1(D_0)$ but not in $S_2(D_0)$ or in $S_3(D_0)$.

$\forall$ n, $p \in D_{n+1} = S(D_n) = S_1(D_n) \cup S_2(D_n) \cup S_3(D_n)$.

$D_n \subset D_o \Rightarrow \quad S_2(D_n) \subset S_2(D_o)$ and $S_3(D_n) \subset S_3(D_o) \underset{p \in S_1(D_o)}{\Rightarrow} p \in S_1(D_n) \quad \forall n.$

$\forall n \; \exists \; q_n \in D_n : \; p = S_1(q_n) = \dfrac{q_n}{2} \Rightarrow q_n = 2p \; \forall \; n \Rightarrow q_n = q_{n+1} =: q \in D$ and

$p = S_1(q) \in S_1(D) \subset S(D) = S_1(D) \cup S_2(D) \cup S_3(D)$.

Remarks

(1) For the closed unit square Q in Example (4) after the definition of a contraction,
$D = \bigcap_{k=0}^\infty S^k(Q)$, i.e. D is the **Sierpinski Right Triangle** .

(2) The Sierpinski Triangle is **not** a topological Cantor set , i.e. not a compact, totally disconnected set in $\Re^2$ without isolated points.

Proof:

D is connected , because all triangle boundaries of D_n belong to D.

Attractor for contractions S_j, $1 \leq j \leq n$, and the iteration of the S_j's.

The attractor for finitely many contractions S_j: X⮌ on a complete metric space X is also the closure of the fixed points of all possible finite compositions of the S_j's. To prove this the following definition will be needed.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Definition

Let (X, d) be a metric space. Then the **diameter** of $B \subset X$, $B \neq \emptyset$ is

$$|B| := \sup\{d(x, y): x, y \in B\}.$$

Remarks

(1) $|B| = 0 \Leftrightarrow \text{card } B = 1$, i.e. B is a singleton.

(2) $A \subset B \Rightarrow |A| \leq |B|$.

(3) The intersection $\cap B_k$ of a decreasing sequence $B_k \supset B_{k+1}$ of nonempty compact sets in

a metric space X is a singleton $\{x\}$ if $\lim |B_k| = 0$.

Furthermore, if $x_k \in B_k$ for all k , then $x_k \to x$.

Proof:
$\cap B_k \in K(X)$, and $|\cap B_k| \leq |B_k|$ for all k. Thus, $\text{card}(\cap B_k) = 1$.

Let $\{x\} = \cap B_k$ and $\varepsilon > 0$. There is an N so that $|B_k| < \varepsilon$ for $N \leq k$

Since $x \in B_k$ for every k,

$$d(x_k, x) \leq |B_k| < \varepsilon \text{ for } N \leq k, \text{ and therefore, } x_n \underset{n\to\infty}{\to} x.$$

(4) If $S: X \circlearrowleft$ is a contraction with contractivity c, then $|S(B)| \leq c \cdot |B|$.

(5) Let $S_1, S_2: X \circlearrowleft$ be contractions and define $S(B) := S_1(B) \cup S_2(B)$ for $B \subset X$.

Then $S^2(B) := S(S(B)) = S(S_1(B)) \cup S(S_2(B))$
$= S_1(S_1(B)) \cup S_2(S_1(B)) \cup S_1(S_2(B)) \cup S_2(S_2(B))$.

Theorem 7 (Hutchinson,1981 (3.1))

Let X be a complete metric space and let S_j: X⮌ be a contraction with contractivity c_j, $1 \leq j \leq n$.

For every map L: $\aleph \to \{1, \ldots, n\}$, $L(k) := L_k$, and for every $B \in K(X)$ with $S_j(B) \subset B$, $1 \leq j \leq n$, denote

$$S_{L_1 \ldots L_k} := S_{L_1} \circ S_{L_2} \circ \ldots \circ S_{L_k}.$$

Then

$$\bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(B)$$

is a singleton, denoted by x_L, and x_L is in the attractor A of the S_j's. Furthermore, x_L is independent of B and defines a surjective map

$$\varphi: \{1, \ldots, n\}^{\aleph} \to A, \qquad L \to x_L,$$

i.e.

$$A = \bigcup_L \left(\bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A) \right).$$

Proof:

(1) $S_{L_1 \ldots L_k, L_{k+1}}(B) \subset S_{L_1 \ldots L_k}(B)$ $\forall k \in \aleph$ is a decreasing sequence of nonempty compact sets and

$$|S_{L_1 \ldots L_k}(B)| \leq c_{L_1} |S_{L_2 \ldots L_k}(B)| \leq \ldots \leq c_{L_1} \cdots c_{L_k} |B| < c^k |B|$$

for $c := \max_{1 \leq j \leq n} c_j < 1$.

After the third remark above,

$$\bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(B) =: \{x_L\}.$$

(2) But $\{x_L\} = \bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A)$, due to $A = \bigcap_{k=1}^{\infty} S^k(B) \subset S(B) := \bigcup_{j=1}^{n} S_j(B) \subset B$ and

Therefore, $S_{L_1 \ldots L_k}(A) \subset S_{L_1 \ldots L_k}(B)$ for all k, implying that $\bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A) \subset \{x_L\}$.

However, $\bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A) \neq \varnothing$, as a deceasing sequence of sets in K(X), and consequently,

$$\bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A) = \{x_L\}.$$

It follows that x_L is independent of B and is in the attractor, because

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

$S(A) = A$ implies $S_j(A) \subset A \ \forall j$.

(3) φ is surjective:

$A = S(A) = \bigcup_{j=1}^{n} S_j(A)$.

If $x \in A$, $\exists L_1 \in \{1, \ldots, n\}$ with $x \in S_{L_1}(A) = S_{L_1}(\bigcup_{j=1}^{n} S_j(A)) = \bigcup_{j=1}^{n} S_{L_1 j}(A) \Rightarrow$

$\Rightarrow \exists L_2 \in \{1, \ldots, n\}$ with $x \in S_{L_1 L_2}(A)$.

By induction, $\forall k \in \aleph \ \exists L_k \in \{1, \ldots, n\}$ with $x \in S_{L_1 \ldots L_k}(A) \Rightarrow$

$x \in \bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A) \quad \Rightarrow \quad \varphi$ is surjective.

Corollary 8

For $L: \aleph \to \{1, \ldots, n\}$ and $k \in \aleph$, let

$$x_k := x_{\tau_k}$$

with

$$\tau_k : \aleph \to \{1, \ldots, n\}$$

the period k sequence

$$L_1 \ldots L_k L_1 \ldots L_k L_1 \ldots L_k \ldots.$$

Then

$$\lim_{k \to \infty} x_k = x_L \qquad \text{and} \qquad S_{L_1 \ldots L_k}(x_k) = x_k,$$

i.e. $x_k \in A$ is the unique fixed point of the contraction $S_{L_1 \ldots L_k}$.

Proof:

(1) $\{x_L\} = \bigcap_{k=1}^{\infty} S_{L_1 \ldots L_k}(A) \subset S_{L_1 \ldots L_k}(A) \qquad \forall k.$

$$\forall k, \quad \{x_k\} = \bigcap_{l=1}^{\infty} S_{\tau_k(1) \ldots \tau_k(l)}(A) \subset S_{\tau_k(1) \ldots \tau_k(l)}(A) \quad \forall l.$$

Set $l := k$. $\Rightarrow x_k \in S_{L_1 \ldots L_k}(A) \quad \forall k \quad \Rightarrow x_k \to x_L$ after the remark above.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

(2) $\forall\, q \in ℵ$ and $j_1, \dots, j_q \in \{1, \dots, n\}$,

$$S_{j_1 \dots j_q}(x_k) = S_{j_1 \dots j_q}\left(\bigcap_{l=1}^{\infty} S_{\tau_{k(1)} \dots \tau_{k(l)}}(A)\right) = \bigcap_{l=1}^{\infty} S_{j_1 \dots j_q \tau_{k(1)} \dots \tau_{k(l)}}(A) = x_{j_1 \dots j_q L_1 \dots L_k L_{k+1} \dots}$$

Now set $q = k$ and $j_l = L_l$ for $1 \le l \le k$. Then $S_{L_1 \dots L_k}(x_k) = x_k$, and

x_k is a fixed point .

Remark

Every fixed point of every contraction $S_{L_1 \dots L_k}$ is in A for all $k \in ℵ$ and all $L: ℵ \to \{1, \dots, n\}$.

Corollary 9

A is the closure of the set of fixed points of the $S_{L_1 \dots L_k}$ $\forall\, L: ℵ \to \{1, \dots, n\}$ and $\forall\, k \in ℵ$.

Proof:

"$\subset$" $\forall\, x \in A\ \exists\, L \in \{1, \dots, n\}^{ℵ}$: $x = x_L = \lim x_k$ by 2.7 and 2.8.

"$\supset$" Every fixed point x_k of $S_{L_1 \dots L_k}$ is in A, and if $x = \lim x_k$, then $x \in A$, since A is closed.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Image Data Compression

Pictures may be worth a thousand words, but they require a lot of computer memory to store which is expensive. Images are stored in computers as bits (a bit is a binary information unit, either 0 or 1, on or off). In 1994 the storage per bit cost a half a millionth of a dollar,

$$1 \text{ bit} \quad = \quad \frac{1}{2 \cdot 10^6} = 0.5 \times 10^{-6} \text{ dollars.}$$

Human eyes can process around 8 millions bits so that a moderate quality photo costs about $4 to store and a photo album about $1000 for 250 images.

Data compression reduces costs and also increases the speed of data transmition, saving time and money.

The Sierpinski Triangle and the Cantor Middle Thirds Set show how a small number of contractions can determine an intricate structure .In this way, finitely many contractions can be used for data compression.

One of the most popular methods of data compression, the so-called JPEG standard (Joint Photographic Experts Group) uses a different approach, namely Fourier Transforms (in particular, the Discrete Cosine Transform) which filters out the high-frequency Fourier coefficients.

Image compression using contractions was commercialized by Michael Barnsley, the author of *"Fractals Everywhere"*, 1988, Academic Press (2nd edition 2000, Morgan Kaufmann)who founded the company Iterated Systems Inc. in Atlanta, Georgia which appears to no longer be in existence.

Data Compression Problems

(1) What compact sets are attractors A or can be approximated by attractors A for finitely many contractions?

(2) How can finitely many contractions S_j approximating a compact set be found? (The encoding problem).

The attractor A is approximated by every iterate $S^k(B)$ for $B \in K(X)$, since $S^k(B) \to A$, and the more iterates the better the approximation if $S(B) \subset B$.

The next theorem, which is a straightforward corollary of the classical Contraction Mapping Principle, gives an error estimate:

Collage Theorem 10

Let X be a complete metric space and let $S_j: X\circlearrowleft$ be contractions with contractivity c_j, $1 \leq j \leq n$.

If $c = \max_{1 \leq j \leq n} c_j$, then for any $B \in K(X)$ and any $k \geq 0$

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

$$d_H(S^k(B), A) \le \frac{c^k}{1-c}\, d_H(B, S(B)).$$

Proof: (By induction)

k=0:

$$d_H(B, A) \le d_H(B, S(B)) + d_H(S(B), A) \qquad \text{[Triangle Inequality]}$$

$$\le d_H(B, S(B)) + c\cdot d_H(B, A),$$

from 2.5 using A = S(A).

$k \Rightarrow k + 1$:

$$d_H(S^{k+1}(B), A) = d_H(S^k(S(B),A) \le \frac{c^k}{1-c}\, d_H(S^2(B), S(B)) \le \frac{c^k}{1-c}\, c\cdot d_H(S(B), B), \quad \text{by 2.5.}$$

Remarks

(1) In the visual arts a *collage* is a collection of pieces cut out from various materials and then pasted together. In this sense, $S(B) = \bigcup_{j=1}^{n} S_j(B)$ is a collage.

(2) The smaller $d_H(B, S(B))$ is, the better the approximation $d_H(B, A)$ of B to A is,

because $d_H(A, B) \le \frac{1}{1-c}\, d_H(B, S(B))$.

(3) The closer c is to 0, the better the approximation of B to A. ($\frac{1}{1-c}>1$).

(4) For **encoding** (i.e. given $B\in K(X)$, find $n\in \aleph$ and find n contractions $S_j:X\circlearrowleft$ whose attractor A resembles B), find contractions S_j so that $S(B) = \bigcup_{j=1}^{n} S_j(B)$ is close to B in the Hausdorff metric. If S_j are similarities, $S_j(A)$ is a homeomorphic copy of A.

Examples

1) Sierpinski Right Triangle

Goal:
Find the contractions $S_j:\Re^2\circlearrowleft$ whose attractor is the Sierpinski Right Triangle.
Method:

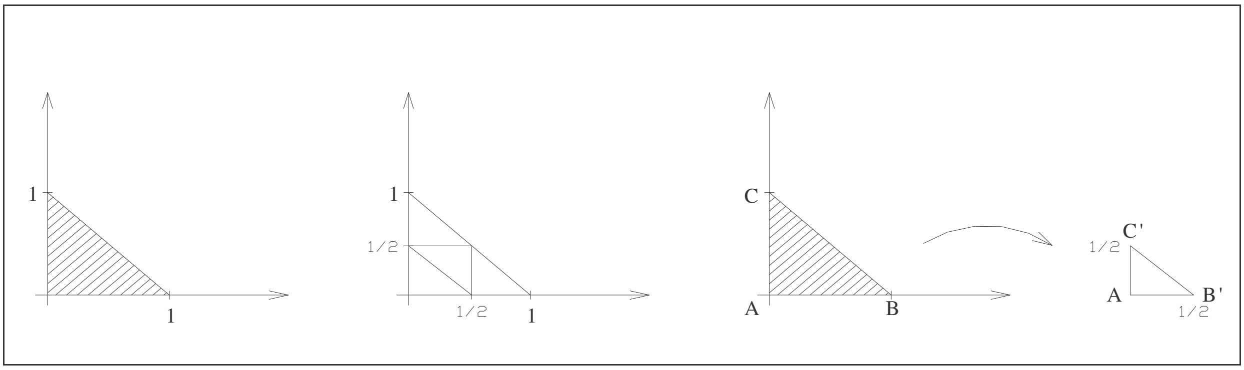
Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

How many reduced affine images are needed to cover the original triangle? Obviously, the answer is three. Find these three affine maps.

An affine map $\Re^2 \circlearrowleft$ is uniquely determined by any 3 points in $\Re^2$ and their images, provided such points are not collinear.

Find $S_1\begin{pmatrix} x \\ y \end{pmatrix} := M\begin{pmatrix} x \\ y \end{pmatrix} + p$, for $p \in \Re^2$ and M a 2x2-matrix, so that $S_1(0, 0) = (0, 0)$

(implying p = 0), $S_1(1, 0) = (\frac{1}{2}, 0)$, and $S_1(0, 1) = (0, \frac{1}{2})$.



S_1

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$\left.\begin{array}{l} ax+by=\alpha \\ cx+dy=\beta \end{array}\right\}$$

4 unknowns a, b, c, d in 6 linear equations (2 equations for each point together with its image)

$(0, 0) \to (0, 0) = (\alpha, \beta)$: $\alpha = \beta = 0$

$(1, 0) \to (\frac{1}{2}, 0) = (\alpha, \beta)$: $a = \frac{1}{2}$, $c = 0$

$(0, 1) \to (0, \frac{1}{2}) = (\alpha, \beta)$: $b = 0$, $d = \frac{1}{2}$

$$\Rightarrow \quad S_1(x, y) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}.$$

S_1 is a contraction if $\|M\|<1$ with

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

$\|M\| := \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|} = \sqrt{\lambda_{max}}$, $x \in \Re^2$. λ_{max} is the maximum eigenvalue of M^tM . In our case,

$$M^tM = M^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad \lambda_{max} = \frac{1}{4}, \quad \sqrt{\lambda_{max}} = \frac{1}{2}.$$

2) If $X = \Re^2$, n = 4, and c = 0.6, then for any B in $\Re^2$ with $d_H(B, S(B)) < 0.02$,

$$d_H(B, A) < 0.05.$$

Note that if $d_H(B,S(B)) < 1.0$, then $d_H(B, A) \leq 2.5$.

Proof:

$d_H(B, A) \leq (1/ 1 – 0.6)\, d_H(B, S(B)) < 0.02/0.4 = 0.05$ in the first case,and in the second case

$$d_H(B, A) \leq 1/1\text{-}0.6 = 1/0.4 = 2.5.$$

Any compact set in $\Re^d$ can be approximated arbitrarily closely by a **self-similar** set, i.e. by an attractor given by contracting similarities, as will be shown next. Beforehand recall that an invertible matrix M is orthogonal if $M^t = M^{-1}$ and that the only linear isometries are given by orthogonal matrices (see for example Fischer, Lineare Algebra).

Theorem 11

S: $\Re^n$⮌ is a similarity if and only if there is an r>0 , a vector $a \in \Re^n$, and an orthogonal nxn-matrix $M \in O(n)$ so that

$$S(x) = r\, M(x) + a,$$

i.e. S is a combination of a translation by a vector a, a scaling by r, and an orthogonal linear transformation ; in particular, S is affine.

Proof: Denote the Euclidean norm of $x \in \Re^n$ by $| x |$. Then $| x |^2 = \langle x, x\rangle$ for the Euclidean inner product.

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

"⇒" :

Let $|S(x) – S(y)| = r\,|x – y| \quad \forall\, x, y \in \Re^n$ with $r > 0$.

Define $T(x) := \frac{1}{r}(S(x) – S(0)) \quad \Rightarrow \; T(0) = 0$, and $|T(x) – T(y)| = \frac{1}{r}|S(x) – S(y)| = |x – y|$,

i.e. T is an isometry fixing 0. In particular, T is norm preserving , i.e. $|T(x)| = |x|$.

T preserves inner products:

$2 <x, y> = |x|^2 + |y|^2 – |x – y|^2 \qquad \forall x, y \qquad (*)$

since by the bilinearity of the inner product $<x-y, x-y> = <x, y> + <y, y> - 2 <x, y>$.

Therefore, $|x – y|^2 = |T(x) – T(y)|^2$,since T is an isometry , and by (*)

$|T(x) – T(y)|^2 = |T(x)|^2 + |T(y)|^2 – 2 <T(x), T(y)>$

$= |x|^2 + |y|^2 – 2 <T(x), T(y)>$, since T preserves the norm.

$\Rightarrow \quad <T(x), T(y)> = <x, y>$ using (*) again. (+)

T is linear:

Let $\{e_j : 1 \le j \le n\}$ be an orthogonal basis for $\Re$, i.e. $<e_i, e_j> = \delta_{ij}$.

$\Rightarrow$ $\{T(e_j): 1 \le j \le n\}$ is also an orthogonal basis, because of (+).

$\Rightarrow T(x) = \sum_{j=1}^{n} < T(x), T(e_j) > T(e_j)$ Fourier series

$= \sum_{j=1}^{n} < x, e_j > T(e_j)$ by (+).

$\Rightarrow$ T is linear .

T is orthogonal, since it is an isometry .

$\Rightarrow T(x) = M(x)$ with $M \in O(n)$

$\Rightarrow r\,T(x) = r\,M(x) = S(x) – S(0)$

$\Rightarrow S(x) = r\,M(x) + S(0)$.

"⇐":

Prof. Dr. Sandra Hayes
hayes@mathematik.tu-muenchen.de

Let S(x) = r M(x) + a.

Then S is a similarity, since $|S(x) - S(y)| = r\,|M(x - y)| = r\,|x - y|$,

because orthogonal matrices preserve inner products and thus also norms.

Examples (Orthogonal Transformations)

$$M := \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

(1) Rotation counterclockwise by ϕ

$$M^T M := \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) Reflection on the y-axis

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$(x, y) \to (-x, y)$, $M^T = M$, $M \cdot M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(3) Reflection on the x-axis

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Theorem 12

Let B be a nonempty compact subset of $\Re^d$. For every $\varepsilon > 0$ there are finitely many contracting similarities $S_1, \ldots S_n: \Re^d$ ⮌ for which the attractor A satisfies

$$d_H(B, A) < \varepsilon .$$